SKINNER-RUSK UNIFIED FORMALISM FOR OPTIMAL CONTROL SYSTEMS AND APPLICATIONS

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Abstract

A geometric approach to time-dependent optimal control problems is proposed. This formulation is based on the Skinner and Rusk formalism for Lagrangian and Hamiltonian systems. The corresponding unified formalism developed for optimal control systems allows us to formulate geometrically the necessary conditions given by a weak form of Pontryagin's Maximum Principle, provided that the differentiability with respect to controls is assumed and the space of controls is open. Furthermore, our method is also valid for implicit optimal control systems and, in particular, for the so-called descriptor systems (optimal control problems including both differential and algebraic equations).

Key words: Lagrangian and Hamiltonian formalisms; jet bundles, implicit optimal control systems, descriptor systems.

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1 Introduction

In 1983 Skinner and Rusk introduced a representation of the dynamics of an autonomous mechanical system which combines the Lagrangian and Hamiltonian features [24]. Briefly, in this formulation, one starts with a differentiable manifold Q as the configuration space, and the Whitney sum $TQ \oplus T^*Q$ as the evolution space (with canonical projections $\rho_1 : TQ \oplus T^*Q \longrightarrow TQ$ and $\rho_2 : TQ \oplus T^*Q \longrightarrow T^*Q$). Define on $TQ \oplus T^*Q$ the presymplectic 2-form $\Omega = \rho_2^*\omega_Q$, where ω_Q is the canonical symplectic form on T^*Q, and observe that the rank of this presymplectic form is everywhere equal to $2n$. If the dynamical system under consideration admits a Lagrangian description, with Lagrangian $L \in C^\infty(TQ)$, then we obtain a (presymplectic)-Hamiltonian representation on $TQ \oplus T^*Q$ given by the presymplectic 2-form Ω and the Hamiltonian function $H = \langle \rho_1, \rho_2 \rangle - \rho_1^* L$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and covectors on Q. In this Hamiltonian system the dynamics is given by vector fields X, which are solutions to the Hamiltonian equation $i(X)\Omega = dH$. If L is regular, then there exists a unique vector field X solution to the previous equation, which is tangent to the graph of the Legendre map $\mathcal{F}L \colon TQ \longrightarrow T^*Q$. In the singular case, it is necessary to develop a constraint algorithm in order to find a submanifold (if it exists) where there exists a well-defined dynamical vector field.

The idea of this formulation was to obtain a common framework for both regular and singular dynamics, obtaining simultaneously the Hamiltonian and Lagrangian formulations of the dynamics. Over the years, however, Skinner and Rusk's framework was extended in many directions. For instance, Cantrijn *et al* [7] extended this formalism for explicit time-dependent systems using a jet bundle language; Cortés *et al* [6] use the Skinner and Rusk formalism to consider vakonomic mechanics and the comparison between the solutions of vakonomic and nonholonomic mechanics. In [9, 13, 21] the authors developed the Skinner-Rusk model for classical field theories.

Furthermore, the Skinner-Rusk formalism seems to be a natural geometric setting for Pontryagin's Maximum Principle. In this paper, whose roots are in the developments made in [7, 9, 13], we adapt the Skinner-Rusk formalism to study time-dependent optimal control problems. In this way we obtain a geometric version of the Maximum Principle that can be applied to a wide range of control systems. For instance, these techniques enables to tackle geometrically implicit optimal control systems, that is, those where the control equations are implicit. In fact, systems of differential-algebraic equations appear frequently in control theory. Usually, in the literature, it is assumed that it is possible to rewrite the problem as an explicit system of differential equations, perhaps using the algebraic conditions to eliminate some variables, as in the case of holonomic constraints. However, in general, a control system is described as a system of equations of the type $F(t, x, \dot{x}, u) = 0$, where the x's denote the state variables and the u's the control variables, and there are some interesting cases where the system is not described by the traditional equations $\dot{x} = G(t, x, u)$.

The main results of this work can be found in Sections 3 and 4, where we give a general method to deal with explicit and implicit systems. As examples, we consider the case of optimal control of Lagrangian mechanical systems (see [1, 2, 3, 4]) and also optimal control for descriptor systems [17, 18]. Both examples have significant engineering applications.

The organization of the paper is as follows: Section 2 is devoted to giving an alternative approach of the Skinner-Rusk formalism for time dependent mechanical systems. In Section 3 we develop the unified formalism for explicit time-dependent optimal control problems giving a geometric Pontryagin's Maximum Principle in a weak form, and in Section 4 we do the same for implicit optimal control systems. Section 5 is devoted to examples and applications: first we study the optimal control of Lagrangian systems with controls; that is, systems defined by a Lagrangian and external forces depending on controls [1, 2, 3, 4]. These are considered

as implicit systems defined by the Euler-Lagrange equations. Second, we analyze a quadratic optimal control problem for a descriptor system [17]. We point out the importance of these kinds of systems in engineering problems [18] and references therein. Finally, we include an Appendix where geometric features about Tulczyjew's operators, contact systems and the Euler-Lagrange equations for forced systems are explained.

All the manifolds are real, second countable and $\mathcal{C}^\infty$. The maps are assumed to be $\mathcal{C}^\infty$. Sum over repeated indices is understood.

2 Skinner-Rusk unified formalism for non-autonomous systems

This formalism is a particular case of the unified formalism for field theories developed in [9] and also in [13]. See [7] for an alternative but equivalent approach, and [11] for an extension of this formalism to other kinds of more general time-dependent singular differential equations.

In the jet bundle description of non-autonomous dynamical systems, the configuration bundle is $\pi\colon E \longrightarrow \mathbb{R}$, where E is a $(n+1)$-dimensional differentiable manifold endowed with local coordinates (t, q^i), and $\mathbb{R}$ has as a global coordinate t. The jet bundle of local sections of π, $J^1\pi$, is the *velocity phase space* of the system, with natural coordinates (t, q^i, v^i), adapted to the bundle $\pi\colon E \longrightarrow \mathbb{R}$, and natural projections

$$\pi^1\colon J^1\pi \longrightarrow E \quad , \quad \bar{\pi}^1\colon J^1\pi \longrightarrow \mathbb{R}\,.$$

A Lagrangian density $\mathcal{L} \in \mathit{\Omega}^1(J^1\pi)$ is a $\bar{\pi}^1$-semibasic 1-form on $J^1\pi$, and it is usually written as $\mathcal{L} = L\,\mathrm{d}t$, where $L \in \mathrm{C}^\infty(J^1\pi)$ is the *Lagrangian function* determined by $\mathcal{L}$. Throughout this paper we denote by $\mathrm{d}t$ the volume form in $\mathbb{R}$, and its pull-backs to all the manifolds.

The canonical structure of the bundle $J^1\pi$ allows us to define the *Poincaré-Cartan forms* associated with the Lagrangian density $\mathcal{L}$, and then the Euler-Lagrange equations are written intrinsically (see [10, 23]).

Furthermore, we have the *extended momentum phase space* T^*E, and the *restricted momentum phase space* which is defined by $J^1\pi^* = \mathrm{T}^*E/\pi^*\mathrm{T}^*\mathbb{R}$. Local coordinates in these manifolds are (t, q^i, p, p_i) and (t, q^i, p_i), respectively. Then, the following natural projections are

$$\tau^1\colon J^1\pi^* \longrightarrow E \quad , \quad \bar{\tau}^1 = \pi\circ\tau^1\colon J^1\pi^* \longrightarrow \mathbb{R} \quad , \quad \mu\colon \mathrm{T}^*E \longrightarrow J^1\pi^* \quad , \quad p\colon \mathrm{T}^*E \longrightarrow \mathbb{R}\,.$$

Let $\Theta \in \mathit{\Omega}^1(\mathrm{T}^*E)$ and $\Omega = -\mathrm{d}\Theta \in \mathit{\Omega}^2(\mathrm{T}^*E)$ be the canonical forms of T^*E whose local expressions are

$$\Theta = p_i\mathrm{d}q^i + p\mathrm{d}t \quad , \quad \Omega = \mathrm{d}q^i\wedge\mathrm{d}p_i + \mathrm{d}t\wedge\mathrm{d}p\,.$$

The Hamilton equations can be written intrinsically from these canonical structures (see, for instance, [10, 12, 16, 20, 22]).

Now we introduce the geometric framework for the unified Skinner-Rusk formalism for non-autonomous systems. We define the *extended jet-momentum bundle* $\mathcal{W}$ and the *restricted jet-momentum bundle* $\mathcal{W}_r$

$$\mathcal{W} = J^1\pi\times_E \mathrm{T}^*E \quad , \quad \mathcal{W}_r = J^1\pi\times_E J^1\pi^*$$

with natural coordinates (t, q^i, v^i, p, p_i) and (t, q^i, v^i, p_i), respectively. We have the natural submersions

$$\begin{aligned} \rho_1\colon \mathcal{W} \longrightarrow J^1\pi \ , \ \rho_2\colon \mathcal{W} \longrightarrow \mathrm{T}^*E \ , \ \rho_E\colon \mathcal{W} \longrightarrow E \ , \ \rho_{\mathbb{R}}\colon \mathcal{W} \longrightarrow \mathbb{R} \\ \rho_1^r\colon \mathcal{W}_r \longrightarrow J^1\pi \ , \ \rho_2^r\colon \mathcal{W}_r \longrightarrow J^1\pi^* \ , \ \rho_E^r\colon \mathcal{W}_r \longrightarrow E \ , \ \rho_{\mathbb{R}}^r\colon \mathcal{W}_r \longrightarrow \mathbb{R}\,. \end{aligned} \tag{1}$$

Note that $\pi^1 \circ \rho_1 = \tau^1 \circ \mu \circ \rho_2 = \rho_E$. In addition, for $\bar{y} \in J^1\pi$, and $\mathbf{p} \in \mathrm{T}^*E$, there is also the natural projection

$$\begin{array}{rccc} \mu_{\mathcal{W}} : & \mathcal{W} & \longrightarrow & \mathcal{W}_r \\ & (\bar{y}, \mathbf{p}) & \mapsto & (\bar{y}, [\mathbf{p}]) \end{array}$$

where $[\mathbf{p}] = \mu(\mathbf{p}) \in J^1\pi^*$. The bundle $\mathcal{W}$ is endowed with the following canonical structures:

Definition 1 *1. The* coupling 1-form *in* $\mathcal{W}$ *is the* $\rho_{\mathbb{R}}$*-semibasic* 1*-form* $\hat{\mathcal{C}} \in \Omega^1(\mathcal{W})$ *defined as follows: for every* $w = (j^1\phi(t), \alpha) \in \mathcal{W}$ *(that is,* $\alpha \in T^*_{\rho_E(w)}E$*) and* $V \in \mathrm{T}_w\mathcal{W}$*, then*

$$\hat{\mathcal{C}}(V) = \alpha(\mathrm{T}_w(\phi \circ \rho_{\mathbb{R}})V) \, .$$

2. The canonical 1-form $\Theta_{\mathcal{W}} \in \Omega^1(\mathcal{W})$ *is the* ρ_E*-semibasic form defined by* $\Theta_{\mathcal{W}} = \rho_2^*\Theta$*.*
The canonical 2-form *is* $\Omega_{\mathcal{W}} = -\mathrm{d}\Theta_{\mathcal{W}} = \rho_2^*\Omega \in \Omega^2(\mathcal{W})$*.*

Being $\hat{\mathcal{C}}$ a $\rho_{\mathbb{R}}$-semibasic form, there is $\hat{C} \in \mathrm{C}^\infty(\mathcal{W})$ such that $\hat{\mathcal{C}} = \hat{C}\mathrm{d}t$. Note also that $\Omega_{\mathcal{W}}$ is degenerate, its kernel being the ρ_2-vertical vectors; then $(\mathcal{W}, \Omega_{\mathcal{W}})$ is a presymplectic manifold.

The local expressions for $\Theta_{\mathcal{W}}$, $\Omega_{\mathcal{W}}$, and $\hat{\mathcal{C}}$ are

$$\Theta_{\mathcal{W}} = p_i \mathrm{d}q^i + p\mathrm{d}t \quad , \quad \Omega_{\mathcal{W}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}p \wedge \mathrm{d}t \quad , \quad \hat{\mathcal{C}} = (p + p_i v^i)\mathrm{d}t \, .$$

Given a Lagrangian density $\mathcal{L} \in \Omega^1(J^1\pi)$, we denote $\hat{\mathcal{L}} = \rho_1^*\mathcal{L} \in \Omega^1(\mathcal{W})$, and we can write $\hat{\mathcal{L}} = \hat{L}\mathrm{d}t$, with $\hat{L} = \rho_1^*L \in \mathrm{C}^\infty(\mathcal{W})$. We define a *Hamiltonian submanifold*

$$\mathcal{W}_0 = \{w \in \mathcal{W} \mid \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w)\} \, .$$

So, $\mathcal{W}_0$ is the submanifold of $\mathcal{W}$ defined by the regular constraint function $\hat{C} - \hat{L} = 0$. Observe that this function is globally defined in $\mathcal{W}$, using the dynamical data and the geometry. In local coordinates this constraint function is

$$p + p_i v^i - \hat{L}(t, q^j, v^j) = 0 \tag{2}$$

and its meaning will be clear when we apply this formalism to Optimal Control problems (see Section 3.2). The natural imbedding is $\jmath_0 \colon \mathcal{W}_0 \hookrightarrow \mathcal{W}$, and we have the projections (submersions), see diagram (3):

$$\rho_1^0 \colon \mathcal{W}_0 \longrightarrow J^1\pi \ , \ \rho_2^0 \colon \mathcal{W}_0 \longrightarrow \mathrm{T}^*E \ , \ \rho_E^0 \colon \mathcal{W}_0 \longrightarrow E \ , \ \rho_{\mathbb{R}}^0 \colon \mathcal{W}_0 \longrightarrow \mathbb{R}$$

which are the restrictions to $\mathcal{W}_0$ of the projections (1), and

$$\hat{\rho}_2^0 = \mu \circ \rho_2^0 \colon \mathcal{W}_0 \longrightarrow J^1\pi^* \, .$$

Local coordinates in $\mathcal{W}_0$ are (t, q^i, v^i, p_i), and we have that

$$\begin{array}{lcl} \rho_1^0(t, q^i, v^i, p_i) = (t, q^i, v^i) & , & \jmath_0(t, q^i, v^i, p_i) = (t, q^i, v^i, L - p_i v^i, p_i) \\ \hat{\rho}_2^0(t, q^i, v^i, p_i) = (t, q^i, p_i) & , & \rho_2^0(t, q^i, v^i, p_i) = (t, q^i, L - p_i v^i, p_i) \, . \end{array}$$

Proposition 1 $\mathcal{W}_0$ *is a* 1*-codimensional* $\mu_{\mathcal{W}}$*-transverse submanifold of* $\mathcal{W}$*, which is diffeomorphic to* $\mathcal{W}_r$*.*

(*Proof*) For every $(\bar{y}, \mathbf{p}) \in \mathcal{W}_0$, we have $L(\bar{y}) \equiv \hat{L}(\bar{y}, \mathbf{p}) = \hat{C}(\bar{y}, \mathbf{p})$, and

$$(\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}, \mathbf{p}) = \mu_{\mathcal{W}}(\bar{y}, \mathbf{p}) = (\bar{y}, \mu(\mathbf{p}))\,.$$

First, $\mu_{\mathcal{W}} \circ \jmath_0$ is injective: let $(\bar{y}_1, \mathbf{p}_1), (\bar{y}_2, \mathbf{p}_2) \in \mathcal{W}_0$, then we have

$$(\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}_1, \mathbf{p}_1) = (\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}_2, \mathbf{p}_2) \;\Rightarrow\; (\bar{y}_1, \mu(\mathbf{p}_1)) = (\bar{y}_2, \mu(\mathbf{p}_2)) \;\Rightarrow\; \bar{y}_1 = \bar{y}_2 \;,\; \mu(\mathbf{p}_1) = \mu(\mathbf{p}_2)$$

hence $L(\bar{y}_1) = L(\bar{y}_2) = \hat{C}(\bar{y}_1, \mathbf{p}_1) = \hat{C}(\bar{y}_2, \mathbf{p}_2)$. In a local chart, the third equality gives

$$p(\mathbf{p}_1) + p_i(\mathbf{p}_1) v^i(\bar{y}_1) = p(\mathbf{p}_2) + p_i(\mathbf{p}_2) v^i(\bar{y}_2)$$

but $\mu(\mathbf{p}_1) = \mu(\mathbf{p}_2)$ implies that

$$p_i(\mathbf{p}_1) = p_i([\mathbf{p}_1]) = p_i([\mathbf{p}_2]) = p_i(\mathbf{p}_2)$$

therefore $p(\mathbf{p}_1) = p(\mathbf{p}_2)$ and hence $\mathbf{p}_1 = \mathbf{p}_2$.

Second, $\mu_{\mathcal{W}} \circ \jmath_0$ is onto, then, if $(\bar{y}, [\mathbf{p}]) \in \mathcal{W}_r$, there exists $(\bar{y}, \mathbf{q}) \in \jmath_0(\mathcal{W}_0)$ such that $[\mathbf{q}] = [\mathbf{p}]$. In fact, it suffices to take $[\mathbf{q}]$ such that, in a local chart of $J^1\pi \times_E \mathrm{T}^*E = \mathcal{W}$

$$p_i(\mathbf{q}) = p_i([\mathbf{p}]) \;,\; p(\mathbf{q}) = L(\bar{y}) - p_i([\mathbf{p}]) v^i(\bar{y})\,.$$

Finally, since $\mathcal{W}_0$ is defined by the constraint function $\hat{C} - \hat{L}$ and, as $\ker \mu_{\mathcal{W}*} = \left\{ \dfrac{\partial}{\partial p} \right\}$ locally and $\dfrac{\partial}{\partial p}(\hat{C} - \hat{L}) = 1$, then $\mathcal{W}_0$ is $\mu_{\mathcal{W}}$-transversal. ■

As a consequence of this result, the submanifold $\mathcal{W}_0$ induces a section of the projection $\mu_{\mathcal{W}}$,

$$\hat{h} \colon \mathcal{W}_r \longrightarrow \mathcal{W}\;.$$

Locally, $\hat{h}$ is specified by giving the local *Hamiltonian function* $\hat{H} = -\hat{L} + p_i v^i$; that is, $\hat{h}(t, q^i, v^i, p_i) = (t, q^i, v^i, -\hat{H}, p_i)$. In this sense, $\hat{h}$ is said to be a *Hamiltonian section* of $\mu_{\mathcal{W}}$.

So we have the following diagram

$$\begin{tikzcd}
& J^1\pi & \\
\mathcal{W}_0 \arrow[ur, "\rho_1^0"] \arrow[r, "\jmath_0"] \arrow[dr, "\rho_2^0"'] \arrow[ddr, "\hat{\rho}_2^0"'] & \mathcal{W} \arrow[u, "\rho_1"'] \arrow[r, "\mu_{\mathcal{W}}"] \arrow[d, "\rho_2"'] & \mathcal{W}_r \arrow[ul, "\rho_1^r"'] \arrow[dl, "\rho_2 \circ \hat{h}"] \arrow[ddl, "\rho_2^r"] \\
& T^*E \arrow[d, "\mu"'] & \\
& J^1\pi^* &
\end{tikzcd} \qquad (3)$$

3 Optimal control theory

3.1 Classical formulation of Pontryagin's Maximum Principle

In this section we consider non-autonomous optimal control systems. This class of systems are determined by the *state equations*, which are a set of differential equations

$$\dot{q}^i = \mathcal{F}^i(t, q^j(t), u^a(t))\,,\; 1 \le i \le n\,, \qquad (4)$$

where t is time, q^j denote the state variables and u^a, $1 \leq a \leq m$, the control inputs of the system that must be determined. Prescribing initial conditions of the state variables and fixing control inputs we know completely the trajectory of the state variables $q^j(t)$ (in the sequel, all the functions are assumed to be at least C^2). The objective is the following:

Statement 1 (Non-autonomous optimal control problem). *Find a C^2-piecewise smooth curve $\gamma(t) = (t, q^j(t), u^a(t))$ and $T \in \mathbb{R}^+$ satisfying the conditions for the state variables at time* 0 *and T, the control equations (4); and minimizing the functional $\mathcal{J}(\gamma) = \int_0^T \mathbb{L}(t, q^j(t), u^a(t))\, \mathrm{d}t$.*

The solutions to this problem are called *optimal trajectories*.

The necessary conditions to obtain the solutions to such a problem are provided by *Pontryagin's Maximum Principle* for non-autonomous systems. In this case, considering the time as another state variable, we have [19]:

Theorem 1 (Pontryagin's Maximum Principle). *If a curve $\gamma : [0,T] \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, $\gamma(t) = (t, q^i(t), u^a(t))$, with $\gamma(0)$ and $\gamma(T)$ fixed, is an optimal trajectory, then there exist functions* $p(t)$, $p_i(t)$, $1 \leq i \leq n$, *verifying:*

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{\partial \mathcal{H}}{\partial p_i}(t, q^i(t), u^a(t), p(t), p_i(t)) && (5) \\ \frac{dp_i}{dt} &= -\frac{\partial \mathcal{H}}{\partial q^i}(t, q^i(t), u^a(t), p(t), p_i(t)) && (6) \\ \mathcal{H}(t, q^i(t), u^a(t), p(t), p_i(t)) &= \max_{u^a} \mathcal{H}(t, q^i(t), u^a, p(t), p_i(t)), \quad t \in [0,T] && (7) \end{aligned}$$

and, moreover,

$$\mathcal{H}(t, q^i(t), u^a(t), p(t), p_i(t)) = 0, \quad t \in [0,T] \; , \tag{8}$$

where

$$\mathcal{H}(t, q^i, u^a, p, p_i) = p + p_j \mathcal{F}^j(t, q^i, u^a) + p_0 \mathbb{L}(t, q^i, u^a)$$

and $p_0 \in \{-1, 0\}$.

When we are looking for extremal trajectories, which are those satisfying the necessary conditions of Theorem 1, condition (7) is usually replaced by the weaker condition

$$\varphi_a \equiv \frac{\partial \mathcal{H}}{\partial u^a} = 0, \quad 1 \leq a \leq m \, . \tag{9}$$

In this weaker form, the Maximum Principle only applies to optimal trajectories with optimal controls interior to the control set.

Remark: An extremal trajectory is called *normal* if $p_0 = -1$ and *abnormal* if $p_0 = 0$. For the sake of simplicity, we only consider normal extremal trajectories, but the necessary conditions for abnormal extremals can also be characterized geometrically using the formalism given in Section 2. Hence, from now on we will take $p_0 = -1$.

An optimal control problem is said to be *regular* if the following matrix has maximal rank

$$\left(\frac{\partial \varphi_a}{\partial u^b} \right) = \left(\frac{\partial^2 \mathcal{H}}{\partial u^a \partial u^b} \right) \; . \tag{10}$$

In the following sections we develop a geometric formulation of this Maximum Principle in its weak form, similar to the Skinner-Rusk approach to non-autonomous mechanics as was explained in Section 2 and references therein.

3.2 Unified geometric framework for optimal control theory

In a global description, we have a fiber bundle structure $\pi^C : C \longrightarrow E$ and $\pi : E \longrightarrow \mathbb{R}$, where E is equipped with natural coordinates (t, q^i) and C is the bundle of controls, with coordinates (t, q^i, u^a).

The state equations can be geometrically described as a smooth map $\mathcal{F} : C \longrightarrow J^1\pi$ such that it makes commutative the following diagram

$$
\begin{tikzcd}
C \arrow[rr, "\mathcal{F}"] \arrow[dr, "\pi^C"] \arrow[ddr, "\bar{\pi}^C"'] & & J^1\pi \arrow[dl, "\pi^1"'] \arrow[ddl, "\bar{\pi}^1"] \\
& E \arrow[d, "\pi"] & \\
& \mathbb{R} &
\end{tikzcd}
$$

which means that $\mathcal{F}$ is a jet field along π^C and also along $\bar{\pi}^C$. Locally we have $\mathcal{F}(t, q^i, u^a) = (t, q^i, \mathcal{F}^i(t, q^i, u^a))$.

Geometrically, we will assume that an *optimal control system* is determined by the pair $(\mathbf{L}, \mathcal{F})$, where $\mathbf{L} \in \mathit{\Omega}^1(C)$ is a $\bar{\pi}^C$-semibasic 1-form, then $\mathbf{L} = \mathbb{L}\mathrm{d}t$, with $\mathbb{L} \in \mathrm{C}^\infty(C)$ representing the cost function; and $\mathcal{F}$ is the jet field introduced in the above section.

In this framework, Theorem 1 in its weak form can be restated as:

Theorem 2 *If a curve $\gamma : I \to C$, with $\gamma(0)$ and $\gamma(T)$ fixed, is an optimal trajectory, then there exists a curve $\Gamma : I \to C \times_E T^*E$ such that, in a natural coordinate system, $\Gamma(t) = (\gamma(t), p(t), p_i(t))$ verifies (5), (6), (8) and (9), where $\mathcal{H} = p + p_j\mathcal{F}^j + p_0\mathbb{L}$ and $p_0 \in \{-1, 0\}$.*

Now, we develop the geometric model of Optimal Control theory according to the Skinner-Rusk formulation.

The graph of the mapping $\mathcal{F}$, $\mathrm{Graph}\,\mathcal{F}$, is a subset of $C \times_E J^1\pi$ and allows us to define the *extended* and the *restricted control-jet-momentum bundles*, respectively:

$$
\mathcal{W}^{\mathcal{F}} = \mathrm{Graph}\,\mathcal{F} \times_E T^*E \quad , \quad \mathcal{W}_r^{\mathcal{F}} = \mathrm{Graph}\,\mathcal{F} \times_E J^1\pi^*
$$

which are submanifolds of $C \times_E \mathcal{W} = C \times_E J^1\pi \times_E \mathrm{T}^*E$ and $C \times_E \mathcal{W}_r = C \times_E J^1\pi \times_E J^1\pi^*$, respectively.

In $\mathcal{W}^{\mathcal{F}}$ and $\mathcal{W}_r^{\mathcal{F}}$ we have natural coordinates (t, q^i, u^a, p, p_i) and (t, q^i, u^a, p_i), respectively. We have the immersions (see diagram (11)):

$$
\begin{array}{rcll}
i^{\mathcal{F}} & : & \mathcal{W}^{\mathcal{F}} \hookrightarrow C \times_E \mathcal{W}\,, & i^{\mathcal{F}}(t, q^i, u^a, p, p_i) = (t, q^i, u^a, \mathcal{F}^i(t, q^j, u^b,), p, p_i) \\
i_r^{\mathcal{F}} & : & \mathcal{W}_r^{\mathcal{F}} \hookrightarrow C \times_E \mathcal{W}_r\,, & i_r^{\mathcal{F}}(t, q^i, u^a, p_i) = (t, q^i, u^a, \mathcal{F}^i(t, q^j, u^b), p_i)\ ,
\end{array}
$$

and taking the natural projection

$$
\sigma_{\mathcal{W}} : C \times_E \mathcal{W} \longrightarrow \mathcal{W}
$$

we can construct the pullback of the coupling 1-form $\hat{\mathcal{C}}$ and of the forms $\Theta_{\mathcal{W}}$ and $\Omega_{\mathcal{W}}$ to $\mathcal{W}^{\mathcal{F}}$:

$$
\mathcal{C}_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^*\hat{\mathcal{C}} \quad , \quad \Theta_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^*\Theta_{\mathcal{W}} \quad , \quad \Omega_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^*\Omega_{\mathcal{W}} = (\rho_2^{\mathcal{F}})^*\Omega\,,
$$

see Definition 1, whose local expressions are:

$$\mathcal{C}_{\mathcal{W}^{\mathcal{F}}} = (p + p_i \mathcal{F}^i(t, q^j, u^a))\mathrm{d}t \quad , \quad \Theta_{\mathcal{W}^{\mathcal{F}}} = p_i \mathrm{d}q^i + p\mathrm{d}t \quad , \quad \Omega_{\mathcal{W}^{\mathcal{F}}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}p \wedge \mathrm{d}t\,.$$

Hence, we can draw the diagram

$$\begin{tikzcd}
C \times_E \mathcal{W} \arrow[rrrr, "\mathrm{Id} \times \mu_{\mathcal{W}}"] \arrow[ddddr, "\sigma_{\mathcal{W}}"'] & & & & C \times_E \mathcal{W}_r \arrow[ddddl, "\sigma_{\mathcal{W}_r}"] \\
& & \mathrm{Graph}\,\mathcal{F} & & \\
& \mathcal{W}^{\mathcal{F}} \arrow[uul, "i^{\mathcal{F}}"'] \arrow[ur] \arrow[rr, "\mu_{\mathcal{W}^{\mathcal{F}}}"] \arrow[dr, "\rho_2^{\mathcal{F}}"'] & & \mathcal{W}_r^{\mathcal{F}} \arrow[ul] \arrow[uur, "i_r^{\mathcal{F}}"'] & \\
& & \mathrm{T}^*E & & \\
& \mathcal{W} \arrow[ur, "\rho_2"] \arrow[rr, "\mu_{\mathcal{W}}"] & & \mathcal{W}_r &
\end{tikzcd} \tag{11}$$

where $\rho_2^{\mathcal{F}}$, ρ_2, $\mu_{\mathcal{W}^{\mathcal{F}}}$, and $\sigma_{\mathcal{W}_r}$ are natural projections.

Furthermore we can define the unique function $H_{\mathcal{W}^{\mathcal{F}}} \colon \mathcal{W}^{\mathcal{F}} \longrightarrow \mathbb{R}$ by the condition

$$\mathcal{C}_{\mathcal{W}^{\mathcal{F}}} - (\rho_1^{\mathcal{F}})^* \mathbf{L} = H_{\mathcal{W}^{\mathcal{F}}} \mathrm{d}t\,.$$

where $\rho_1^{\mathcal{F}} \colon \mathcal{W}^{\mathcal{F}} \longrightarrow C$ is another natural projection. This function $H_{\mathcal{W}^{\mathcal{F}}}$ is locally described as

$$H_{\mathcal{W}^{\mathcal{F}}}(t, q^i, u^a, p, p_i) = p + p_i \mathcal{F}^i(t, q^j, u^a) - \mathbb{L}(t, q^j, u^a)\,; \tag{12}$$

(compare this expression with (2)). This is the natural Pontryagin Hamiltonian function as appears in Theorem 1.

Let $\mathcal{W}_0^{\mathcal{F}}$ be the submanifold of $\mathcal{W}^{\mathcal{F}}$ defined by the vanishing of $H_{\mathcal{W}^{\mathcal{F}}}$; that is,

$$\mathcal{W}_0^{\mathcal{F}} = \{ w \in \mathcal{W}^{\mathcal{F}} \mid H_{\mathcal{W}^{\mathcal{F}}}(w) = 0 \}\,.$$

In local coordinates, $\mathcal{W}_0^{\mathcal{F}}$ is given by the constraint

$$p + p_i \mathcal{F}^i(t, q^j, u^a) - \mathbb{L}(t, q^j, u^a) = 0\ .$$

Observe that, in this way, we recover the condition (8). An obvious set of coordinates in $\mathcal{W}_0^{\mathcal{F}}$ is (t, q^i, u^a, p_i). We denote by $\jmath_0^{\mathcal{F}} \colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow \mathcal{W}^{\mathcal{F}}$ the natural embedding; in local coordinates,

$$\jmath_0^{\mathcal{F}}(t, q^i, u^a, p_i) = (t, q^i, u^a, \mathbb{L}(t, q^j, u^b) - p_i \mathcal{F}^i(t, q^j, u^b), p_j)\ .$$

In a similar way to Proposition 1, we may prove the following:

Proposition 2 *$\mathcal{W}_0^{\mathcal{F}}$ is a 1-codimensional $\mu_{\mathcal{W}^{\mathcal{F}}}$-transverse submanifold of $\mathcal{W}^{\mathcal{F}}$, diffeomorphic to $\mathcal{W}_r^{\mathcal{F}}$.*

As a consequence, the submanifold $\mathcal{W}_0^{\mathcal{F}}$ induces a section of the projection $\mu_{\mathcal{W}^{\mathcal{F}}}$,

$$\hat{h}^{\mathcal{F}} \colon \mathcal{W}_r^{\mathcal{F}} \longrightarrow \mathcal{W}^{\mathcal{F}}\ . \tag{13}$$

Locally, $\hat{h}^{\mathcal{F}}$ is specified by giving the local *Hamiltonian function* $\hat{H}^{\mathcal{F}} = p_j \mathcal{F}^j - \mathbb{L}$; that is, $\hat{h}^{\mathcal{F}}(t, q^i, u^a, p_i) = (t, q^i, u^a, p = -\hat{H}^{\mathcal{F}}, p_i)$. The map $\hat{h}^{\mathcal{F}}$ is called a *Hamiltonian section* of $\mu_{\mathcal{W}^{\mathcal{F}}}$.

Thus, we can draw the diagram, where all the projections are natural

$$
\begin{tikzcd}
 & & J^1\pi \arrow[dll, "\pi^1"'] \arrow[drr, "\bar{\pi}^1"] & & \\
E & & C \arrow[ll, "\pi^C"'] \arrow[rr, "\bar{\pi}^C"] \arrow[u, "\mathcal{F}"] & & \mathbb{R} \\
\mathcal{W}_0^{\mathcal{F}} \arrow[u, "\rho_E^{0\mathcal{F}}"] \arrow[urr, "\rho_1^{0\mathcal{F}}"] \arrow[rr, "\jmath_0^{\mathcal{F}}"] \arrow[drr, "\rho_2^{0\mathcal{F}}"] \arrow[ddrr, "\hat{\rho}_2^{0\mathcal{F}}"'] & & \mathcal{W}^{\mathcal{F}} \arrow[ull, "\rho_E^{\mathcal{F}}"] \arrow[u, "\rho_1^{\mathcal{F}}"] \arrow[urr, "\rho_{\mathbb{R}}^{\mathcal{F}}"] \arrow[rr, "\mu_{\mathcal{W}^{\mathcal{F}}}"] \arrow[d, "\rho_2^{\mathcal{F}}"] & & \mathcal{W}_r^{\mathcal{F}} \arrow[ull, "\rho_1^{r\mathcal{F}}"] \arrow[u, "\rho_{\mathbb{R}}^{r\mathcal{F}}"'] \arrow[dll, "\rho_2^{r\mathcal{F}} \circ \hat{h}^{\mathcal{F}}"'] \arrow[ddll, "\rho_2^{r\mathcal{F}}"] \\
 & & T^*E \arrow[d, "\mu"] & & \\
 & & J^1\pi^* & &
\end{tikzcd}
\tag{14}
$$

Finally we define the forms

$$
\Theta_{\mathcal{W}_0^{\mathcal{F}}} = (\jmath_0^{\mathcal{F}})^* \Theta_{\mathcal{W}^{\mathcal{F}}} \quad , \quad \Omega_{\mathcal{W}_0^{\mathcal{F}}} = (\jmath_0^{\mathcal{F}})^* \Omega_{\mathcal{W}^{\mathcal{F}}}
$$

with local expressions

$$
\Theta_{\mathcal{W}_0^{\mathcal{F}}} = p_i \mathrm{d}q^i + (\mathbb{L} - p_i \mathcal{F}^i)\mathrm{d}t \quad , \quad \Omega_{\mathcal{W}_0^{\mathcal{F}}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}(\mathbb{L} - p_i \mathcal{F}^i) \wedge \mathrm{d}t \, .
$$

3.3 Optimal Control equations

Now we are going to establish the dynamical problem for the system $(\mathcal{W}_0^{\mathcal{F}}, \Omega_{\mathcal{W}_0^{\mathcal{F}}})$ and as a consequence we obtain a geometrical version of the weak form of the Maximum Principle.

Proposition 3 *Let $(\mathbf{L}, \mathcal{F})$ define a regular optimal control problem, then there exists a submanifold $\mathcal{W}_1^{\mathcal{F}}$ of $\mathcal{W}_0^{\mathcal{F}}$ and a unique vector field $Z \in \mathfrak{X}(\mathcal{W}_0^{\mathcal{F}})$ tangent to $\mathcal{W}_1^{\mathcal{F}}$ such that*

$$
[i(Z)\Omega_{\mathcal{W}_0^{\mathcal{F}}}]|_{\mathcal{W}_1^{\mathcal{F}}} = 0 \quad , \quad [i(Z)\mathrm{d}t]|_{\mathcal{W}_1^{\mathcal{F}}} = 1 \; . \tag{15}
$$

The integral curves Γ of Z satisfy locally the necessary conditions of Theorem 2.

(*Proof*) In a natural coordinate system, we have

$$
Z = f \frac{\partial}{\partial t} + A^i \frac{\partial}{\partial q^i} + B^a \frac{\partial}{\partial u^a} + C_i \frac{\partial}{\partial p_i}
$$

where f, A^i, B^a, C_i are unknown functions in $\mathcal{W}_0^{\mathcal{F}}$. Then, the second equation (15) leads to $f = 1$, and from the first we obtain that

$$\text{coefficients in } \mathrm{d}p_i: \quad \mathcal{F}^i - A^i = 0 \tag{16}$$

$$\text{coefficients in } \mathrm{d}u^a: \quad \frac{\partial \mathbb{L}}{\partial u^a} - p_j \frac{\partial \mathcal{F}^j}{\partial u^a} = 0 \tag{17}$$

$$\text{coefficients in } \mathrm{d}q^i: \quad \frac{\partial \mathbb{L}}{\partial q^i} - p_j \frac{\partial \mathcal{F}^j}{\partial q^i} - C_i = 0 \tag{18}$$

$$\text{coefficients in } \mathrm{d}t: \quad -A^i \frac{\partial \mathbb{L}}{\partial q^i} + A^i p_j \frac{\partial \mathcal{F}^j}{\partial q^i} - B^a \frac{\partial \mathbb{L}}{\partial u^a} + B^a p_j \frac{\partial \mathcal{F}^j}{\partial u^a} + C_i \mathcal{F}^i = 0\,. \tag{19}$$

Now, if $\Gamma(t) = (t, q^i(t), u^a(t), p_i(t))$ is an integral curve of Z, we have that $A^i = \dfrac{dq^i}{dt}$, $B^a = \dfrac{du^a}{dt}$, $C_i = \dfrac{dp_i}{dt}$.

The *Pontryagin Hamiltonian function* is $\mathcal{H} = p + p_i \mathcal{F}^i - \mathbb{L}$. As we are in $\mathcal{W}_0^{\mathcal{F}}$, condition (8), $\mathcal{H} = 0$, is satisfied. Furthermore,

- From (16) we deduce that $A^i = \mathcal{F}^i$; that is, $\dfrac{dq^i}{dt} = \dfrac{\partial \mathcal{H}}{\partial p_i}$, which are the equations (5).
- Equations (17) determine a new set of conditions
$$\varphi_a = \frac{\partial \mathbb{L}}{\partial u^a} - p_j \frac{\partial \mathcal{F}^j}{\partial u^a} = \frac{\partial \mathcal{H}}{\partial u^a} = 0 \tag{20}$$
which are equations (9). We assume that they define the new submanifold $\mathcal{W}_1^{\mathcal{F}}$ of $\mathcal{W}_0^{\mathcal{F}}$. We denote by $\jmath_1^{\mathcal{F}} \colon \mathcal{W}_1^{\mathcal{F}} \hookrightarrow \mathcal{W}_0^{\mathcal{F}}$ the natural embedding.
- From (18) we completely determine the functions $C_i = \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{H}}{\partial q^i}$; which are the equations (6).
- Finally, using (16), (18) and (17) it is easy to prove that equations (19) hold identically.

Furthermore Z must be tangent to $\mathcal{W}_1^{\mathcal{F}}$, that is,

$$Z(\varphi_a) = Z\left(\frac{\partial \mathcal{H}}{\partial u^a}\right) = 0 \qquad (\text{on } \mathcal{W}_1^{\mathcal{F}})$$

or, in other words,

$$0 = \frac{\partial^2 \mathcal{H}}{\partial t \partial u^a} + \mathcal{F}^i \frac{\partial^2 \mathcal{H}}{\partial q^i \partial u^a} + B^b \frac{\partial^2 \mathcal{H}}{\partial u^b \partial u^a} - \frac{\partial \mathcal{H}}{\partial q^i} \frac{\partial^2 \mathcal{H}}{\partial p_i \partial u^a} \qquad (\text{on } \mathcal{W}_1^{\mathcal{F}})\,. \tag{21}$$

However, as the optimal control problem is regular, the matrix $\dfrac{\partial^2 \mathcal{H}}{\partial u^b \partial u^a}$ has maximal rank. Then the equations (21) determine all the coefficients B^b. ■

As a direct consequence of this proposition, we state the intrinsic version of Theorem 2.

Theorem 3 (Geometric weak Pontryagin's Maximum Principle). *If $\gamma \colon I \to C$ is a solution to the regular optimal control problem given by $(\mathbf{L}, \mathcal{F})$, then there exists an integral curve of a vector field $Z \in \mathfrak{X}(\mathcal{W}_0^{\mathcal{F}})$, whose projection to C is γ, and such that Z is a solution to the equations*

$$i(Z)\Omega_{\mathcal{W}_0^{\mathcal{F}}} = 0 \quad , \quad i(Z)\mathrm{d}t = 1\ ,$$

in a submanifold $\mathcal{W}_1^{\mathcal{F}}$ of $\mathcal{W}_0^{\mathcal{F}}$, which is given by the condition (20).

Note that the conditions fulfilled by the integral curves of Z, satisfying the suitable initial conditions, imply that their natural projections on C are γ.

Remark: In fact, the second equation of (15) could be relaxed to the condition

$$i(Z)\mathrm{d}t \neq 0 \ ,$$

which determines vector fields transversal to π whose integral curves are equivalent to those obtained above, with arbitrary reparametrization.

Note that, using the implicit function theorem on the equations $\varphi_a = 0$, we get the functions $u^a = u^a(t, q, p)$. Therefore, for regular control problems, we can choose local coordinates (t, q^i, p_i) on $\mathcal{W}_1^{\mathcal{F}}$, and $\mathcal{H}|_{\mathcal{W}_1^{\mathcal{F}}}$ is locally a function of these coordinates.

If the control problem is not regular, then one has to implement a constraint algorithm to obtain a final constraint submanifold $\mathcal{W}_f^{\mathcal{F}}$ (if it exists) where the vector field Z is tangent (see, for instance, [8]).

Let $\jmath_1 \colon \mathcal{W}_1^{\mathcal{F}} \to \mathcal{W}_0^{\mathcal{F}}$ be the natural embedding, the form $\Omega_{\mathcal{W}_1^{\mathcal{F}}} = (\jmath_1^{\mathcal{F}})^*\Omega_{\mathcal{W}_0^{\mathcal{F}}}$ is locally written as

$$\Omega_{\mathcal{W}_1^{\mathcal{F}}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}\mathcal{H}|_{\mathcal{W}_1^{\mathcal{F}}} \wedge \mathrm{d}t \, .$$

Hence, for optimal control problems, taking into account the regularity of the matrix (10), we have the following:

Proposition 4 *If the optimal control problem is regular, then* $(\mathcal{W}_1^{\mathcal{F}}, \Omega_{\mathcal{W}_1^{\mathcal{F}}}, \mathrm{d}t)$ *is a cosymplectic manifold, that is,* $(\Omega_{\mathcal{W}_1^{\mathcal{F}}})^n \wedge \mathrm{d}t$ *is a volume form (see [15]).*

4 Implicit optimal control problems

4.1 Unified geometric framework for implicit optimal control problems

The formalism presented in Section 3.2 is valid for a more general class of optimal control problems not previously considered from a geometric perspective: optimal control problems whose state equations are *implicit*, that is,

$$\Psi^\alpha(t, q, \dot{q}, u) = 0 \ , \ 1 \leq \alpha \leq s \ , \text{ with } \mathrm{d}\Psi^1 \wedge \ldots \wedge \mathrm{d}\Psi^s \neq 0 \, . \tag{22}$$

There are several examples of these kinds of optimal control problems, some of them coming from engineering applications. In Section 5 we study two specific examples: the descriptor systems which appear in electrical engineering and the controlled Lagrangian systems which play a relevant role in robotics.

From a more geometric point of view, we may interpret Equations (22) as constraint functions determining a submanifold M_C of $C \times_E J^1\pi$, with natural embedding $\jmath^{M_C} \colon M_C \hookrightarrow C \times_E J^1\pi$. We will also assume that $(\pi^C \times \pi^1) \circ \jmath^{M_C} \colon M_C \longrightarrow E$ is a surjective submersion.

In this situation, the techniques presented in the previous section are still valid. Now the implicit optimal control system is determined by the data $(\mathbf{L}, M_C)$, where $\mathbf{L} \in \Omega^1(M_C)$ is a semibasic form with respect to the projection $\tau^{M_C} \colon M_C \longrightarrow \mathbb{R}$, and hence it can be written as $\mathbf{L} = \mathbb{L}\mathrm{d}t$, for some $\mathbb{L} \in \mathrm{C}^\infty(M_C)$. First define the *extended control-jet-momentum manifold* and the *restricted control-jet-momentum manifold*

$$\mathcal{W}^{M_C} = M_C \times_E T^*E \quad , \quad \mathcal{W}_r^{M_C} = M_C \times_E J^1\pi^*$$

which are submanifolds of $C \times_E \mathcal{W} = C \times_E J^1\pi \times_E \mathrm{T}^*E$ and $C \times_E \mathcal{W}_r = C \times_E J^1\pi \times_E J^1\pi^*$, respectively.

We have the canonical immersions (embeddings)

$$i^{M_C} \colon \mathcal{W}^{M_C} \hookrightarrow C \times_E \mathcal{W} \quad , \quad i_r^{M_C} \colon \mathcal{W}_r^{M_C} \hookrightarrow C \times_E \mathcal{W}_r \ .$$

So we can draw a diagram analogous to (11) replacing the core of the diagram by

$$\begin{tikzcd} & M_C & \\ \mathcal{W}^{M_C} \arrow[ur, "\rho_1^{M_C}"] \arrow[rr, "\mu_{\mathcal{W}^{M_C}}"] & & \mathcal{W}_r^{M_C} \arrow[ul, "\rho_1^{rM_C}"'] \end{tikzcd}$$

where all the projections are natural.

Now, consider the pullback of the coupling 1-form $\hat{\mathcal{C}}$ and the forms $\sigma_{\mathcal{W}}^*\Theta_{\mathcal{W}}$ and $\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}}$ to $\mathcal{W}^{M_C}$ by the map $i^{M_C} \colon \mathcal{W}^{M_C} \longrightarrow C \times_E \mathcal{W}$; that is

$$\mathcal{C}_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}} \circ i^{M_C})^*\hat{\mathcal{C}} \ , \ \Theta_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}} \circ i^{M_C})^*\Theta_{\mathcal{W}} \ , \ \Omega_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}} \circ i^{M_C})^*\Omega_{\mathcal{W}} \ ,$$

and denote by $\hat{C} \in \mathrm{C}^\infty(\mathcal{W}^{M_C})$ the unique function such that $\mathcal{C}_{\mathcal{W}^{M_C}} = \hat{C}\mathrm{d}t$. Finally, let $H_{\mathcal{W}^{M_C}} \colon \mathcal{W}^{M_C} \longrightarrow \mathbb{R}$ be the unique function such that $\mathcal{C}_{\mathcal{W}^{M_C}} - (\rho_1^{M_C})^*\mathbf{L} = H_{\mathcal{W}^{M_C}}\mathrm{d}t$. Observe that $H_{\mathcal{W}^{M_C}} = \hat{C} - \hat{\mathbb{L}}$, where $\hat{\mathbb{L}} = (\rho_1^{M_C})^*\mathbb{L}$, and remember that $H_{\mathcal{W}^{M_C}}$ is the Pontryagin Hamiltonian function, see (12).

Let $\mathcal{W}_0^{M_C}$ be the submanifold of $\mathcal{W}^{M_C}$ defined by the vanishing of $H_{\mathcal{W}^{M_C}}$, i.e.

$$\mathcal{W}_0^{M_C} = \{w \in \mathcal{W}^{M_C} \mid H_{\mathcal{W}^{M_C}}(w) = (\hat{C} - \hat{\mathbb{L}})(w) = 0\} \ , \tag{23}$$

and denote by $\jmath_0^{M_C} \colon \mathcal{W}_0^{M_C} \hookrightarrow \mathcal{W}^{M_C}$ the natural embedding. As in Proposition 1 we may prove the following:

Proposition 5 *$\mathcal{W}_0^{M_C}$ is a 1-codimensional $\mu_{\mathcal{W}^{M_C}}$-transverse submanifold of $\mathcal{W}^{M_C}$, diffeomorphic to $\mathcal{W}_r^{M_C}$.*

As a consequence, the submanifold $\mathcal{W}_0^{\mathcal{F}}$ induces a section of the projection $\mu_{\mathcal{W}^{M_C}}$,

$$\hat{h}^{M_C} \colon \mathcal{W}_r^{M_C} \longrightarrow \mathcal{W}^{M_C} \ .$$

Then we can draw the following diagram, which is analogous to (14), where all the projections

are natural

$$
\begin{tikzcd}
 & C\times_E J^1\pi \arrow[dl] \arrow[dr] & \\
E & M_C \arrow[l, "\pi^{M_C}"'] \arrow[u, "\jmath^{M_C}"] \arrow[r, "\bar{\pi}^{M_C}"] & \mathbb{R} \\
\mathcal{W}_0^{M_C} \arrow[u, "\rho_{_E}^{0M_C}"] \arrow[ur, "\rho_1^{0M_C}"] \arrow[r, "\jmath_0^{M_C}"] \arrow[dr, "\rho_2^{0M_C}"] \arrow[ddr, "\hat{\rho}_2^{0M_C}"'] & \mathcal{W}^{M_C} \arrow[ul, "\rho_{_E}^{M_C}"'] \arrow[u, "\rho_1^{M_C}"] \arrow[ur, "\rho_{_\mathbb{R}}^{M_C}"] \arrow[r, "\mu_{_{\mathcal{W}^{M_C}}}"] \arrow[d, "\rho_2^{M_C}"'] & \mathcal{W}_r^{M_C} \arrow[ul, "\rho_1^{rM_C}"'] \arrow[u, "\rho_{_\mathbb{R}}^{rM_C}"'] \arrow[dl, "\rho_2^{rM_C}\circ\hat{h}^{M_C}"'] \arrow[ddl, "\rho_2^{rM_C}"] \\
 & T^*E \arrow[d, "\mu"'] & \\
 & J^1\pi^* &
\end{tikzcd}
$$

Finally, we define the forms

$$\Theta_{\mathcal{W}_0^{M_C}} = (\jmath_0^{M_C})^*\Theta_{\mathcal{W}^{M_C}} \quad , \quad \Omega_{\mathcal{W}_0^{M_C}} = (\jmath_0^{M_C})^*\Omega_{\mathcal{W}^{M_C}} \ .$$

4.2 Optimal Control equations

Now, we will see how the dynamics of the optimal control problem $(\mathbf{L}, M_C)$ is determined by the solutions (where they exist) of the equations

$$i(Z)\Omega_{\mathcal{W}_0^{M_C}} = 0 \quad , \quad i(Z)\mathrm{d}t = 1 \quad , \quad \text{for } Z \in \mathfrak{X}(\mathcal{W}_0^{M_C}) \ . \tag{24}$$

As in Section 3.3 , the second equation of (24) can be relaxed to the condition

$$i(Z)\mathrm{d}t \neq 0 \ .$$

In order to work in local coordinates we need the following proposition, whose proof is obvious:

Proposition 6 *For a given $w \in \mathcal{W}_0^{M_C}$, the following conditions are equivalent:*

1. *There exists a vector $Z_w \in T_w\mathcal{W}_0^{M_C}$ verifying that*

$$\Omega_{\mathcal{W}_0^{M_C}}(Z_w, Y_w) = 0 \ , \ \text{for every } Y_w \in T_w\mathcal{W}_0^{M_C} \ .$$

2. *There exists a vector $Z_w \in T_w(C \times_E \mathcal{W})$ verifying that*

   (i) $Z_w \in T_w\mathcal{W}_0^{M_C}$,

(ii) $i(Z_w)(\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}})_w \in (T_w\mathcal{W}_0^{M_C})^0$.

As a consequence of this last proposition, we can obtain the implicit optimal control equations using condition 2 as follows: there exists $Z \in \mathfrak{X}(C \times_E \mathcal{W})$ such that

(i) Z is tangent to $\mathcal{W}_0^{M_C}$.

(ii) The 1-form $i(Z)\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}}$ is null on the vector fields tangent to $\mathcal{W}_0^{M_C}$.

As $\mathcal{W}_0^{M_C}$ is defined in (23), and the constraints are $\Psi^\alpha = 0$ and $\hat{C} - \hat{\mathbb{L}} = 0$; then there exist $\lambda_\alpha, \lambda \in \mathrm{C}^\infty(C \times_E \mathcal{W})$, to be determined, such that

$$(i(Z)\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}})|_{\mathcal{W}_0^{M_C}} = (\lambda_\alpha \mathrm{d}\Psi^\alpha + \lambda \mathrm{d}(\hat{C} - \hat{\mathbb{L}}))|_{\mathcal{W}_0^{M_C}} \ .$$

As usual, the undetermined functions λ_α's and λ are called Lagrange multipliers.

Now using coordinates $(t, q^i, u^a, v^i, p, p^i)$ in $C \times_E \mathcal{W}$, we look for a vector field

$$Z = \frac{\partial}{\partial t} + A^i \frac{\partial}{\partial q^i} + B^a \frac{\partial}{\partial u^a} + C^i \frac{\partial}{\partial v^i} + D_i \frac{\partial}{\partial p_i} + E \frac{\partial}{\partial p} \ ,$$

where A^i, B^a, C^i, D_i, E are unknown functions in $\mathcal{W}_0^{M_C}$ verifying the equation

$$\begin{aligned} 0 &= i_Z\left(\mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}t \wedge \mathrm{d}p\right) - \lambda_\alpha \mathrm{d}\Psi^\alpha - \lambda \mathrm{d}(p + p_i v^i - \mathbb{L}(t, q, u)) \\ &= \left(-E - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial t} + \lambda \frac{\partial \mathbb{L}}{\partial t}\right) \mathrm{d}t + \left(\lambda \frac{\partial \mathbb{L}}{\partial q^i} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial q^i} - D_i\right) \mathrm{d}q^i \\ &\quad + \left(\lambda \frac{\partial \mathbb{L}}{\partial u^a} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial u^a}\right) \mathrm{d}u^a + \left(-\lambda p_i - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial v^i}\right) \mathrm{d}v^i \\ &\quad +(A^i - \lambda v^i)\mathrm{d}p_i + (1 - \lambda)\mathrm{d}p \ . \end{aligned}$$

Thus, we obtain $\lambda = 1$, and

$$A^i = v^i \ , \ D_i = \frac{\partial \mathbb{L}}{\partial q^i} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial q^i} \ , \ E = \frac{\partial \mathbb{L}}{\partial t} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial t} \ , \ p_i = -\lambda_\alpha \frac{\partial \Psi^\alpha}{\partial v^i} \ , \ 0 = \frac{\partial \mathbb{L}}{\partial u^a} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial u^a}$$

together with the tangency conditions

$$\begin{aligned} 0 &= Z(\Psi^\alpha)|_{\mathcal{W}_0^{M_C}} = \left(\frac{\partial \Psi^\alpha}{\partial t} + A^i \frac{\partial \Psi^\alpha}{\partial q^i} + B^a \frac{\partial \Psi^\alpha}{\partial u^a} + C^i \frac{\partial \Psi^\alpha}{\partial v^i}\right)\Big|_{\mathcal{W}_0^{M_C}} \\ 0 &= Z(p + p_i v^i - \mathbb{L}(t, q, u))|_{\mathcal{W}_0^{M_C}} \ . \end{aligned}$$

Therefore the equations of motion are:

$$\begin{aligned} \frac{d}{dt}\left(\lambda_\alpha(t) \frac{\partial \Psi^\alpha}{\partial v^i}(t, q(t), \dot{q}(t), u(t))\right) + \frac{\partial \mathbb{L}}{\partial q^i}(t, q(t), u(t)) - \lambda_\alpha(t) \frac{\partial \Psi^\alpha}{\partial q^i}(t, q(t), \dot{q}(t), u(t)) &= 0 \\ \frac{\partial \mathbb{L}}{\partial u^a}(t, q(t), u(t)) - \lambda_\alpha(t) \frac{\partial \Psi^\alpha}{\partial u^a}(t, q(t), \dot{q}(t), u(t)) &= 0 \\ \Psi^\alpha(t, q(t), \dot{q}(t), u(t)) &= 0 \end{aligned}$$

Let $\mathbb{L}_0 = \mathbb{L} - \lambda_\alpha \Psi^\alpha$ be the classical extended Lagrangian for constrained systems. Then these last equations are the usual dynamical equations in optimal control obtained by applying the Lagrange multipliers method to the constrained variational problem, that is, the Euler-Lagrange equations for $\mathbb{L}_0$, the extremum necessary condition at interior points, and the constraints.

Remarks:

- In the particular case that $\Psi^j = v^j - \mathcal{F}^j = 0$, the vector field Z so-obtained is just the image of the vector field obtained in Section 3.3 by the Hamiltonian section (13), as a simple calculation in coordinates shows.

- Another obvious but significant remark is that we can take $\bar{\pi}^k \colon J^k\pi \longrightarrow \mathbb{R}$ (the bundle of k-jets of π) instead of $\pi \colon E \longrightarrow \mathbb{R}$, and hence $J^k\bar{\pi}^k$ and $\mathrm{T}^*J^k\pi$ instead of $J^1\bar{\pi}^1$ and T^*E, respectively. These changes allows us to address those optimal control problems where we have $\Phi^{kC} \colon C \longrightarrow J^k\pi$; that is, we deal with higher-order equations, and their solutions must satisfy that $(\gamma(t), j^{k+1}(\pi^k \circ \Phi^{kC} \circ \gamma)(t)) \in M$, where M is a submanifold of $C \times_{J^k\pi} J^{k+1}\pi$.

5 Applications and examples

5.1 Optimal Control of Lagrangian systems with controls

See Appendix A for previous geometric concepts which are needed in this section. For a complete study of these systems see [2, 4] and references therein.

Now we provide a definition of a *controlled-force*, which allows dependence on time, configuration, velocities and control inputs. In a global description, one assumes a fiber bundle structure $\Phi^{1C} : C \longrightarrow J^1\pi$, where C is the bundle of controls, with coordinates (t, q, v, u). Then a controlled-force is a smooth map $\mathbf{F} : C \longrightarrow \mathcal{C}_\pi$, so that $\pi_{J^1\pi} \circ \mathbf{F} = \Phi^{1C}$ (see diagram (34)).

In a natural chart, a controlled-force is represented by

$$\mathbf{F}(t, q, v, u) = \mathbf{F}_i(t, q, v, u)(\mathrm{d}q^i - v^i \mathrm{d}t) \ .$$

A *controlled Lagrangian system* is defined as the pair $(\mathcal{L}, \mathbf{F})$ which determines an implicit control system described by the subset D_C of $C \times_{J^1\pi} J^2\pi$:

$$\begin{aligned} D_C &= \{(c, \hat{p}) \in C \times_{J^1\pi} J^2\pi \mid (\imath_1^* d_T \Theta_{\mathcal{L}} - (\pi_1^2)^* \mathrm{d}L)(\hat{p}) = ((\pi_1^2)^* \mathbf{F})(c)\} \\ &= \{(c, \hat{p}) \in C \times_{J^1\pi} J^2\pi \mid \mathcal{E}_{\mathcal{L}}(\hat{p}) = ((\pi_1^2)^* \mathbf{F})(c)\} \\ &= \{(c, \hat{p}) \in C \times_{J^1\pi} J^2\pi \mid (\mathcal{E}_{\mathcal{L}} \circ pr_2 - (\pi_1^2)^* \mathbf{F} \circ pr_1)(c, \hat{p}) = 0\} \end{aligned}$$

where pr_1 and pr_2 are the natural projections from $C \times_{J^1\pi} J^2\pi$ onto the factors. In fact, D_C is not necessarily a submanifold of $C \times_{J^1\pi} J^2\pi$. There are a lot of cases where this does happen. In local coordinates

$$\begin{aligned} D_C = \Bigg\{ & (t, q, v, w, u) \in C \times_{J^1\pi} J^2\pi \ \Bigg| \ \frac{\partial^2 L}{\partial v^i \partial v^j}(t, q, v) w^j + \frac{\partial^2 L}{\partial v^i \partial q^j}(t, q, v) v^j \\ & + \frac{\partial^2 L}{\partial v^i \partial t}(t, q, v) - \frac{\partial L}{\partial q^i}(t, q, v) - \mathbf{F}_i(t, q, v, u) = 0 \Bigg\} \ . \end{aligned}$$

A solution to the controlled Lagrangian system $(\mathcal{L}, \mathbf{F})$ is a map $\gamma \colon \mathbb{R} \longrightarrow C$ satisfying that:

(i) $\Phi^{1C} \circ \gamma = j^1(\pi^1 \circ \Phi^{1C} \circ \gamma)$.

(ii) $(\gamma(t), j^2(\pi^1 \circ \Phi^{1C} \circ \gamma)(t)) \in D_C$, for every $t \in \mathbb{R}$.

The condition (i) means that $\Phi^{1C}\circ\gamma$ is holonomic, and (ii) is the condition (35) of Appendix A.3; that is, the Euler-Lagrange equations for the controlled Lagrangian system $(\mathcal{L},\mathbf{F})$.

Now, consider the map $(\mathrm{Id},\Upsilon)\colon C\times_{J^1\pi}J^2\pi\longrightarrow C\times_{J^1\pi}J^1\bar{\pi}^1$, where $\Upsilon\colon J^2\pi\longrightarrow J^1\bar{\pi}^1$ is defined in (33) (see Appendix A.2), and let $M_C=(\mathrm{Id},\Upsilon)(D_C)$. As (Id,Υ) is an injective map, we can identify $D_C\subset C\times_{J^1\pi}J^2\pi$ with this subset M_C of $C\times_{J^1\pi}J^1\bar{\pi}^1$. Observe that there is a natural projection from M_C to $J^1\pi$.

If $\mathbb{L}\colon M_C\longrightarrow\mathbb{R}$ is a cost function, we may consider the implicit optimal control system determined by the pair $(\mathbf{L},M_C)$, where $\mathbf{L}=\mathbb{L}\mathrm{d}t$, and apply the method developed in Section 4.

Let $\overline{\mathcal{W}}^{M_C}=M_C\times_{J^1\pi}T^*J^1\pi$, and $\overline{\mathcal{W}}^C=C\times_{J^1\pi}J^1\bar{\pi}^1\times_{J^1\pi}T^*J^1\pi$. The natural projection from $\overline{\mathcal{W}}^C$ to $T^*J^1\pi$ allows us to pull-back the canonical 2-form $\Omega_{J^1\pi}$ to a presymplectic form $\Omega_{\overline{\mathcal{W}}^C}\in\mathit{\Omega}^2(\overline{\mathcal{W}}^C)$. Furthermore, in $J^1\bar{\pi}^1\times_{J^1\pi}T^*J^1\pi$ there is the natural coupling form $\hat{\bar{\mathcal{C}}}$ (see Definition 1). We denote by $\bar{\mathcal{C}}$ its pull-back to $\overline{\mathcal{W}}^C$. We denote by $\mathbf{L}$ and $\mathbb{L}$ the pull-back of $\mathbf{L}$ and $\mathbb{L}$ from M_C to $\overline{\mathcal{W}}^C$, for the sake of simplicity.

Then, let $\bar{H}_{\mathcal{W}^C}\colon\overline{\mathcal{W}}^C\longrightarrow\mathbb{R}$ be the unique function such that $\bar{\mathcal{C}}-\mathbf{L}=\bar{H}_{\mathcal{W}^C}\mathrm{d}t$, whose local expression is $\bar{H}_{\mathcal{W}^C}=p+p_i\bar{v}^i+\bar{p}_iw^i-\mathbb{L}$, and consider the submanifold $\overline{\mathcal{W}}_0=\{\tilde{q}\in\overline{\mathcal{W}}^C\mid\bar{H}_{\mathcal{W}^C}(\tilde{q})=0\}$. The pull-back of $\bar{H}_{\mathcal{W}^C}$ to $\overline{\mathcal{W}}^{M_C}$ is the Pontryagin Hamiltonian, denoted by $\bar{H}_{\mathcal{W}^{M_C}}$.

Finally, the dynamics is in the submanifold $\overline{\mathcal{W}}_0^{M_C}=\overline{\mathcal{W}}^{M_C}\cap\overline{\mathcal{W}}_0$ of $\overline{\mathcal{W}}^C$, where $\jmath_1^{M_C}$ is the natural embedding. $\overline{\mathcal{W}}_0^{M_C}$ is endowed with the presymplectic form $\Omega_{\overline{\mathcal{W}}_0^{M_C}}=(\jmath_1^{M_C})^*\Omega_{\overline{\mathcal{W}}^C}$. Therefore, the motion is determined by a vector field $Z\in\mathfrak{X}(\overline{\mathcal{W}}_0^{M_C})$ satisfying the equations

$$i(Z)\Omega_{\overline{\mathcal{W}}_0^{M_C}}=0\quad,\quad i(Z)\mathrm{d}t=1\ .$$

A local chart in $\overline{\mathcal{W}}^C$ is $(t,q^i,v^i,\bar{v}^i,w^i,u^a,p,p_i,\bar{p}_i)$, where $(\bar{v}^i,w^i)$ and $(p,p_i,\bar{p}_i)$ are the natural fiber coordinates in $J^1\bar{\pi}^1$ and $\mathrm{T}^*J^1\pi$, respectively. The manifold $\overline{\mathcal{W}}^{M_C}$ is given locally by the $2n$ constraints:

$$\begin{aligned}\varphi_i(t,q^i,v^i,\bar{v}^i,w^i,u^a,p,p_i,\bar{p}_i)&=w^j\frac{\partial^2L}{\partial v^i\partial v^j}(t,q,v)+\bar{v}^j\frac{\partial^2L}{\partial v^i\partial q^j}(t,q,v)+\frac{\partial^2L}{\partial v^i\partial t}(t,q,v)\\&\quad-\frac{\partial L}{\partial q^i}(t,q,v)-\mathbf{F}_i(t,q,v,u)=0\\\bar{\varphi}^i(t,q^i,v^i,\bar{v}^i,w^i,u^a,p,p_i,\bar{p}_i)&=v^i-\bar{v}^i=0\ ,\end{aligned}$$

and $\overline{\mathcal{W}}_0$ is given by

$$\phi(t,q^i,v^i,\bar{v}^i,w^i,u^a,p,p_i,\bar{p}_i)=\bar{H}_{\mathcal{W}^C}(t,q^i,v^i,\bar{v}^i,w^i,u^a,p,p_i,\bar{p}_i)=p+p_i\bar{v}^i+\bar{p}_iw^i-\mathbb{L}(t,q,v,u)=0\ ,$$

and

$$\Omega_{\overline{\mathcal{W}}_0^{M_C}}=\mathrm{d}q^i\wedge\mathrm{d}p_i+\mathrm{d}v^i\wedge\mathrm{d}\bar{p}_i+\mathrm{d}t\wedge\mathrm{d}(\mathbb{L}-p_i\bar{v}^i-\bar{p}_iw^i)\ .$$

Following Proposition 6, we look for a vector field $Z\in\mathfrak{X}(\overline{\mathcal{W}}^C)$ such that, for every $\mathbf{w}\in\overline{\mathcal{W}}_0^{M_C}$:

$$\text{(i)}\quad Z_{\mathbf{w}}\in\mathrm{T}_{\mathbf{w}}\overline{\mathcal{W}}_0^{M_C}\quad,\quad\text{(ii)}\quad i(Z_{\mathbf{w}})\Omega_{\overline{\mathcal{W}}^C}\in(\mathrm{T}_{\mathbf{w}}\overline{\mathcal{W}}_0^{M_C})^0\ ,$$

or, equivalently

(i) $(\jmath_1^{M_C})^*(Z(\varphi_i))=0,\quad(\jmath_1^{M_C})^*(Z(\bar{\varphi}^i))=0,\quad(\jmath_1^{M_C})^*(Z(\phi))=0.$

(ii) $(j_1^{M_C})^*(i(Z)\Omega_{\overline{\mathcal{W}}^C}) = 0$.

Remember that the constraints are $\varphi_i = 0, \quad \bar{\varphi}^i = 0, \quad \phi = 0$.

If Z is given locally by

$$Z = \frac{\partial}{\partial t} + A^i\frac{\partial}{\partial q^i} + \mathcal{A}^i\frac{\partial}{\partial v^i} + \bar{A}^i\frac{\partial}{\partial \bar{v}^i} + \bar{\mathcal{A}}^i\frac{\partial}{\partial w^i} + B^a\frac{\partial}{\partial u^a} + D\frac{\partial}{\partial p} + C_i\frac{\partial}{\partial p_i} + \bar{C}_i\frac{\partial}{\partial \bar{p}_i} \ ,$$

then $A^i, \mathcal{A}^i, \bar{A}^i, \bar{\mathcal{A}}^i, B^a, D, C_i, \bar{C}_i$ are unknown functions in $\overline{\mathcal{W}}^C$, such that

$$i(Z)\Omega_{\overline{\mathcal{W}}^C} = \lambda^i \mathrm{d}\varphi_i + \bar{\lambda}_i \mathrm{d}\bar{\varphi}^i + \lambda \mathrm{d}(p + p_i\bar{v}^i + \bar{p}_i w^i - \mathbb{L}(t,q,v,u))$$

and $Z(\varphi_i) = 0$, $Z(\bar{\varphi}^i) = 0$ and $Z(p + p_i\bar{v}^i + \bar{p}_i w^i - \mathbb{L}(t,q,v,u)) = 0$. From these equations we obtain

$$\begin{array}{lll} \lambda = 1 \ , & A^i = \bar{v}^i \ , & \mathcal{A}^i = w^i \\ C_i = \dfrac{\partial \mathbb{L}}{\partial q^i} - \lambda^j\dfrac{\partial \varphi_j}{\partial q^i} \ , & \bar{C}_i = \dfrac{\partial \mathbb{L}}{\partial v^i} - \lambda^j\dfrac{\partial \varphi_j}{\partial v^i} - \bar{\lambda}_i \ , & D = \dfrac{\partial \mathbb{L}}{\partial t} - \lambda^j\dfrac{\partial \varphi_j}{\partial t} \\ 0 = \dfrac{\partial \mathbb{L}}{\partial u^a} + \lambda^i\dfrac{\partial \mathbf{F}_i}{\partial u^a} \ , & p_i = \bar{\lambda}_i - \lambda^j\dfrac{\partial^2 L}{\partial v^j\partial q^i} \ , & \bar{p}_i = -\lambda^j\dfrac{\partial^2 L}{\partial v^i\partial v^j} \end{array} \tag{25}$$

and the tangency conditions

$$\begin{aligned} Z(\varphi_i) &= \frac{\partial \varphi_i}{\partial t} + \bar{v}^j\frac{\partial \varphi_i}{\partial q^j} + w^j\frac{\partial \varphi_i}{\partial v^j} + \bar{A}^j\frac{\partial^2 L}{\partial v^i\partial q^j} - B^a\frac{\partial \mathbf{F}_i}{\partial u^a} + \bar{\mathcal{A}}^j\frac{\partial^2 L}{\partial v^i\partial v^j} = 0 \qquad (26)\\ Z(\bar{\varphi}^i) &= w^i - \bar{A}^i = 0 \\ Z(\phi) &= Z(p + p_i\bar{v}^i + \bar{p}_i w^i - \mathbb{L}(t,q,v,u)) = 0 \end{aligned}$$

where the third condition is satisfied identically using the previous equations.

Assuming that the Lagrangian L is regular, that is, $\det(W_{ij}) = det\left(\frac{\partial^2 L}{\partial v^i\partial v^j}\right) \neq 0$, then from equations for p_i and $\bar{p}_i$ in (25) we obtain explicit values of the Lagrange multipliers λ^i and $\bar{\lambda}_i$. Therefore, the remaining equations (25) are now rewritten as the new set of constraints

$$\psi^a(t,q,v,u,\bar{p}) = \frac{\partial \mathbb{L}}{\partial u^a} - W^{ij}\bar{p}_i\frac{\partial \mathbf{F}_j}{\partial u^a} = 0 \ , \tag{27}$$

which corresponds to $\dfrac{\partial \bar{H}_{\mathcal{W}^{MC}}}{\partial u^a} = 0$.

The new compatibility condition is

$$Z(\psi^a) = \frac{\partial \psi^a}{\partial t} + \bar{v}^j\frac{\partial \psi^a}{\partial q^j} + w^j\frac{\partial \psi^a}{\partial v^j} + B^b\frac{\partial \psi^a}{\partial u^b} + \bar{C}_i\frac{\partial \psi^a}{\partial \bar{p}_i} = 0 \ . \tag{28}$$

Furthermore we assume that

$$\det\left(\frac{\partial \psi^a}{\partial u^b}\right) \neq 0 \ ,$$

then, from Equations (26) and (28) we obtain the remaining components $\bar{\mathcal{A}}^i$ and B^a, and we determine completely the vector field Z.

The equations of motion for a curve are determined by the system of implicit-differential equations:

$$\begin{aligned}
\dot p_i(t) &= \frac{\partial \mathbb{L}}{\partial q^i}(t,q(t),\dot q(t),u(t)) - \lambda^j(t,q(t),\dot q(t),\bar p(t))\frac{\partial \varphi_j}{\partial q^i}(t,q(t),\dot q(t),\ddot q(t),u(t)) \\
\dot{\bar p}_i(t) &= \frac{\partial \mathbb{L}}{\partial v^i}(t,q(t),\dot q(t),u(t)) - p_i(t) \\
&\quad -\lambda^j(t,q(t),\dot q(t),\bar p(t))\left[\frac{\partial \varphi_j}{\partial v^i}(t,q(t),\dot q(t),\ddot q(t),u(t)) + \frac{\partial^2 L}{\partial v^j \partial q^i}(t,q(t),\dot q(t))\right] \qquad (29) \\
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(t,q(t),\dot q(t))\right) - \frac{\partial L}{\partial q^i}(t,q(t),\dot q(t)) - \mathbf{F}_i(t,q(t),\dot q(t),u(t)) \qquad (30) \\
0 &= \frac{\partial \mathbb{L}}{\partial u^a}(t,q(t),\dot q(t),u(t)) - W^{ij}(t,q(t),\dot q(t))\bar p_i(t)\frac{\partial \mathbf{F}_j}{\partial u^a}(t,q(t),\dot q(t),u(t)) \ . \qquad (31)
\end{aligned}$$

Equation (31) is the explicit expression of (27).

In [1] the authors study optimal control of Lagrangian systems with controls in a more restrictive situation using higher-order dynamics, obtaining that the states are determined by a set of fourth-order differential equations. First it is necessary to assume that the system is *fully actuated*, that is $m = n$, and $\operatorname{rank}(\Xi_{ij}) = \operatorname{rank}\left(\frac{\partial \mathbf{F}_i}{\partial u^j}\right) = n$. Moreover, in the sequel we assume that the system is affine on controls, that is,

$$\mathbf{F}_i(t,q,\dot q,u) = A_i(t,q,\dot q) + A_{ij}(t,q,\dot q)u^j \ .$$

Therefore, $\Xi_{ij} = A_{ij}$.

Then from the constraint equations (30) and (31), applying the Implicit Function Theorem, we deduce that

$$\begin{aligned}
u^i(t) &= u^i(t,q(t),\dot q(t),\ddot q(t)) = A^{ij}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^j}(t,q(t),\dot q(t))\right) - \frac{\partial L}{\partial q^j}(t,q(t),\dot q(t)) - A_j(t,q(t),\dot q(t))\right] \\
\bar p_i(t) &= \mathcal{H}_i^j(t,q(t),\dot q(t))\frac{\partial \mathbb{L}}{\partial u^j}(t,q(t),\dot q(t),u(t,q(t),\dot q(t),\ddot q(t)))
\end{aligned}$$

where $(\mathcal{H}_i^j)$ are the components of the inverse matrix of the regular matrix $(W^{ik}A_{kj})$.

Taking the derivative with respect to time of Equation (29), and substituting the value of $\dot p_i(t)$ using Equation (29) we obtain a fourth-order differential equation depending on the states. After some computations we deduce that

$$\mathcal{H}_i^j(t,q(t),\dot q(t))\frac{\partial^2 \mathbb{L}}{\partial u^j \partial u^k}(t,q(t),\dot q(t),\ddot q(t))\frac{d^4 q^k}{dt^4}(t) = G_i(t,q(t),\dot q(t),\ddot q(t),\dddot q(t)) \ .$$

Finally, under the assumption that the matrix $\left(\frac{\partial^2 \mathbb{L}}{\partial u^j \partial u^k}\right)$ is invertible, we obtain a explicit fourth-order system of differential equations:

$$\frac{d^4 q^i}{dt^4}(t) = \bar G^i(t,q(t),\dot q(t),\ddot q(t),\dddot q(t)) \ .$$

5.2 Optimal Control problems for descriptor systems

See [17] for the origin and interest of this example. The study of these kinds of systems was suggested to us by Professor. A.D. Lewis (Queen's University of Canada).

Consider the problem of minimizing the functional

$$\mathcal{J} = \frac{1}{2}\int_0^{+\infty} \left[a_i(q^i)^2 + ru^2\right] \,\mathrm{d}t,$$

$1 \leq i \leq 3$, with control equations

$$\dot{q}^2 = q^1 + b_1 u \quad , \quad \dot{q}^3 = q^2 + b_2 u \quad , \quad 0 = q^3 + b_3 u$$

with parameters $a_i, b_i \geq 0$ and $r > 0$.

As in the previous section, the geometric framework developed in Section 3.2 is also valid for this class of systems. Let $E = \mathbb{R} \times \mathbb{R}^3$ with coordinates (t, q^i), and $C = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ with coordinates (t, q^i, u). The submanifold $M_C \subset C \times_E J^1\pi$ is given by

$$M_C = \{(t, q^1, q^2, q^3, v^1, v^2, v^3, u) \mid v^2 = q^1 + b_1 u \;, v^3 = q^2 + b_2 u \;, 0 = q^3 + b_3 u\}\,.$$

The cost function is

$$\begin{array}{rrcl} \mathbb{L}: & C & \longrightarrow & \mathbb{R} \\ & (t, q^1, q^2, q^3, u) & \longmapsto & \frac{1}{2}\left[a_1(q^1)^2 + a_2(q^2)^2 + a_3(q^3)^2 + ru^2\right] \end{array}$$

We analyze the dynamics of the implicit optimal control system determined by the pair $(\mathbf{L}, M_C)$.

Let $\mathcal{W}^{M_C} = M_C \times_E T^*E$ and $\mathcal{W}^C = C \times_E J^1\pi \times_E T^*E$ with coupling form $\mathcal{C}$ inherited from the natural coupling form in $J^1\pi \times T^*E$. Let $H_{\mathcal{W}^C} \colon \mathcal{W}^C \longrightarrow \mathbb{R}$ be the unique function such that $\mathcal{C} - \mathbf{L} = H_{\mathcal{W}^C}\mathrm{d}t$, and consider the submanifold $\mathcal{W}_0 = \{\tilde{q} \in \mathcal{W}^C \mid H_{\mathcal{W}^C}(\tilde{q}) = 0\}$. Finally, the dynamics is in the submanifold $\mathcal{W}_0^{M_C} = \mathcal{W}^{M_C} \cap \mathcal{W}_0$ of $\mathcal{W}^C$. Locally,

$$\begin{aligned} \mathcal{W}_0^{M_C} \;=\; & \{(t, q^1, q^2, q^3, v^1, v^2, v^3, u, p, p_1, p_2, p_3) \mid v^2 = q^1 + b_1 u \;, v^3 = q^2 + b_2 u \;, \\ & q^3 + b_3 u = 0 \;, p + p_1 v^1 + p_2 v^2 + p_3 v^3 - \mathbb{L} = 0\}\,. \end{aligned}$$

Therefore, the motion is determined by a vector field $Z \in \mathfrak{X}(\mathcal{W}_0^{M_C})$ satisfying the Equations (24), which according to Proposition 6 is equivalent to finding a vector field $Z \in \mathfrak{X}(\mathcal{W}^C)$ (if it exists):

$$Z = \frac{\partial}{\partial t} + A^1\frac{\partial}{\partial q^1} + A^2\frac{\partial}{\partial q^2} + A^3\frac{\partial}{\partial q^3} + C^1\frac{\partial}{\partial v^1} + C^2\frac{\partial}{\partial v^2} + C^3\frac{\partial}{\partial v^3} + B\frac{\partial}{\partial u} + D_1\frac{\partial}{\partial p_1} + D_2\frac{\partial}{\partial p_2} + D_3\frac{\partial}{\partial p_3} + E\frac{\partial}{\partial p}$$

such that

$$\begin{gathered} i(Z)\Omega_{\mathcal{W}^C} = \lambda_1 d(q^1 + b_1 u - v^2) + \lambda_2 d(q^2 + b_2 u - v^3) + \lambda_3 d(q^3 + b_3 u) + \lambda dH_{\mathcal{W}^C} \;, \\ Z(q^1 + b_1 u - v^2) = 0 \;, \quad Z(q^2 + b_2 u - v^3) = 0 \;, \quad Z(q^3 + b_3 u) = 0 \;, \quad Z(H_{\mathcal{W}^C}) = 0 \end{gathered}$$

where $\Omega_{\mathcal{W}^C} \in \Omega^2(\mathcal{W}^C)$ is the 2-form with local expression

$$\Omega_{\mathcal{W}^C} = \mathrm{d}q^1 \wedge \mathrm{d}p_1 + \mathrm{d}q^2 \wedge \mathrm{d}p_2 + \mathrm{d}q^3 \wedge \mathrm{d}p_3 + \mathrm{d}t \wedge \mathrm{d}p\,.$$

After some straightforward computations, we obtain that

$$\begin{array}{rcccl} A^1 = v^1 & , & A^2 = q^1 + b_1 u & , & A^3 = q^2 + b_2 u \\ \lambda = 1 & , & E = 0 & , & 0 = ru - b_1 p_2 - b_2 p_3 - b_3 \lambda_3 \\ C^2 = v^1 + b_1 B & , & C^3 = A^2 + b_2 B & , & 0 = A^3 + b_3 B \\ p_1 = 0 & , & p_2 = \lambda_1 & , & p_3 = \lambda_2 \\ D_1 = a_1 q_1 - p_2 & , & D_2 = a_2 q_2 - p_3 & , & D_3 = a_3 q_3 - \lambda_3\,. \end{array}$$

We deduce that

$$\lambda_3 = \frac{1}{b_3}(ru - b_1p_2 - b_2p_3)\,, \quad B = -\frac{1}{b_3}(q^2 + b_2u)\,.$$

Therefore, the new constraint submanifold $\mathcal{W}_1^{M_C} \hookrightarrow \mathcal{W}_0^{M_C}$ is

$$\mathcal{W}_1^{M_C} = \{(t, q^1, q^2, v^1, u, p_1, p_2, p_3) \mid p_1 = 0\}\,.$$

Consistency of the dynamics implies that

$$0 = Z(p_1) = D_1 = a_1q_1 - p_2\,.$$

Thus,

$$\mathcal{W}_2^{M_C} = \{(t, q^1, q^2, v^1, u, p_2, p_3) \mid a_1q_1 - p_2 = 0\}$$

and once again we impose the tangency to the new constraints:

$$0 = Z(a_1q_1 - p_2) = a_1v^1 - a_2q_2 + p_3$$

which implies that

$$\mathcal{W}_3^{M_C} = \{(t, q^1, q^2, v^1, u, p_3) \mid a_1v^1 - a_2q^2 + p_3 = 0\}\,.$$

From the compatibility condition

$$0 = Z(a_1v^1 - a_2q^2 + p_3)$$

and the constraints we determine the remaining component C^1 of Z:

$$C^1 = \frac{1}{a_1b_3}\left[(a_2b_3 - a_1b_1)q^1 - b_2a_2q^2 + (a_2b_1b_3 + a_3b_3^2 + r)u + b_2a_1v^1\right]\,.$$

Therefore the equations of motion of the optimal control problem are:

$$\begin{aligned}
\ddot{q}^1(t) &= \frac{1}{a_1b_3}\left[(a_2b_3 - a_1b_1)q^1(t) - a_2b_2q^2(t) + (a_2b_1b_3 + a_3b_3^2 + r)u(t) + a_1b_2\dot{q}^1(t)\right] \\
\dot{q}^2(t) &= q^1(t) + b_1u(t) \\
0 &= q^2(t) + b_2u(t) - b_3\dot{u}(t)\,.
\end{aligned} \tag{32}$$

From (32) we deduce that

$$u(t) = \frac{1}{a_2b_1b_3 + a_3b_3^2 + r}\left[(a_1b_1 - a_2b_3)q^1(t) + a_2b_2q^2(t) - a_1b_2\dot{q}^1(t) + a_1b_3\ddot{q}^1(t)\right]\,.$$

This is the result obtained in Müller [17], where the optimal feedback control depends on the state variables and also on their derivatives (non-casuality).

Choosing local coordinates (t, q^1, q^2, v^1, u) on $\mathcal{W}_3^{M_C}$, if $\jmath_3 : \mathcal{W}_3^{M_C} \mapsto \mathcal{W}^C$ is the canonical embedding, then $\Omega_{\mathcal{W}_3^{M_C}} = \jmath_3^*\Omega_{\mathcal{W}^C}$ is locally written as

$$\Omega_{\mathcal{W}_3^{M_C}} = -a_1\mathrm{d}q^1 \wedge \mathrm{d}q^2 + a_2b_3\mathrm{d}q^2 \wedge \mathrm{d}u - a_1b_3\mathrm{d}v^1 \wedge \mathrm{d}u + \mathrm{d}t \wedge \mathrm{d}\jmath_3^*p\,,$$

where $\jmath_3^*p : \mathcal{W}_3^{M_C} \longrightarrow \mathbb{R}$ is the function

$$\jmath_3^*p = -\frac{1}{2}a_1(q^1)^2 - \frac{1}{2}a_2(q^2)^2 + \frac{1}{2}(r + a_3b_3^2)u^2 - a_1b_1q^1u - a_2b_2q^2u + a_1b_2v^1u + a_1q^2v^1\,.$$

Obviously, $(\Omega_{\mathcal{W}_3^{M_C}}, \mathrm{d}t)$ is a cosymplectic structure on $\mathcal{W}_3^{M_C}$ (see Proposition 4), and there exists a unique vector field $\bar{Z} \in \mathfrak{X}(\mathcal{W}_3^{M_C})$ satisfying

$$i(\bar{Z})\Omega_{\mathcal{W}_3^{M_C}} = 0, \quad i(\bar{Z})\mathrm{d}t = 1\,.$$

6 Conclusions and outlook

In this paper we have elucidated the geometrical structure of optimal control problems using a variation of the Skinner-Rusk formalism for mechanical systems. The geometric framework allows us to find the dynamical equations of the problem (equivalent to the Pontryagin Maximum Principle for smooth enough problems without boundaries on the space of controls), and to describe the submanifold (if it exists) where the solutions of the problem are consistently defined. The method admits a nice extension for studying the dynamics of implicit optimal control problems with a wide range of applicability.

One line of future research appears when we combine our geometric method for optimal control problems, and the study of the (approximate) solutions to optimal control problems involving partial differential equations when we discretize the space domain and consider the resultant set of ordinary differential equations (see, for instance, [5] and references therein and [14], for a geometrical description). This resultant system is an optimal control problem, where the state equations are, presumably, a very large set of coupled ordinary differential equations. Typically, difficulties other than computational ones appear because the system is differential-algebraic, and therefore the optimal control problem is a usual one for a descriptor system.

Moreover, in this paper we have confined ourselves to the geometrical aspects of time-dependent optimal control problems. Of course, the techniques are suitable for studying the formalism for optimal control problems for partial differential equations in general.

A Appendix

A.1 Tulczyjew's operators

Given a differentiable manifold Q and its tangent bundle $\tau_Q\colon \mathrm{T}Q \longrightarrow Q$, we consider the following operators, introduced by Tulczyjew [25]: first we have $i_T\colon \Omega^k(Q) \longrightarrow \Omega^{k-1}(\mathrm{T}Q)$, which is defined as follows: for every $(\mathrm{p}, v) \in \mathrm{T}Q$, $\alpha \in \Omega^k(Q)$, and $X_1, \ldots, X_{k-1} \in \mathfrak{X}(\mathrm{T}Q)$,

$$(i_T\,\alpha)((\mathrm{p}, v); X_1, \ldots, X_{k-1}) = \alpha(\mathrm{p}; v, \mathrm{T}_{(\mathrm{p},v)}\tau_Q((X_1)_{(\mathrm{p},v)}), \ldots, \mathrm{T}_{(\mathrm{p},v)}\tau_Q((X_{k-1})_{(\mathrm{p},v)}))\ .$$

Then, the so-called *total derivative* is a map $\mathrm{d}_T\colon \Omega^k(Q) \longrightarrow \Omega^k(\mathrm{T}Q)$ defined by

$$d_T = \mathrm{d}\circ i_T + i_T \circ \mathrm{d}\ .$$

For the case $k = 1$, using natural coordinates in $\mathrm{T}Q$, the local expression is

$$d_T\alpha \equiv \mathrm{d}_T(A_j \mathrm{d}q^j) = A_j \mathrm{d}v^j + v^i \frac{\partial A_j}{\partial q^i}\mathrm{d}q^j\ .$$

A.2 Some geometrical structures

Recall that, associated with every jet bundle $J^1\pi$, we have the *contact system*, which is a subbundle $\mathcal{C}_\pi$ of $T^*J^1\pi$ whose fibres at every $j^1\phi(t) \in J^1\pi$ are defined as

$$\mathcal{C}_\pi(j^1\phi(t)) = \{\alpha \in \mathrm{T}^*_{j^1\phi(t)}(J^1\pi) \mid \alpha = (\mathrm{T}_{j^1\phi(t)}\pi^1 - \mathrm{T}_{j^1\phi(t)}(\phi\circ\bar{\pi}^1))^*\beta,\ \beta \in \mathrm{V}^*_{\phi(t)}\pi\}\ .$$

One may readily see that a local basis for the sections of this bundle is given by $\{\mathrm{d}q^i - v^i\mathrm{d}t\}$.

Now, denote by $J^2\pi$ the bundle of 2-jets of π. This jet bundle is equipped with natural coordinates (t, q^i, v^i, w^i) and canonical projections

$$\pi_1^2 \colon J^2\pi \longrightarrow J^1\pi, \quad \pi^2 \colon J^2\pi \longrightarrow E \quad , \quad \bar{\pi}^2 \colon J^2\pi \longrightarrow \mathbb{R} \; .$$

Considering the bundle $J^1\bar{\pi}^1$, we introduce the canonical injection $\Upsilon \colon J^2\pi \longrightarrow J^1\bar{\pi}^1$ given by

$$\Upsilon(j^2\phi(t)) = (j^1(j^1\phi))(t) \, . \tag{33}$$

Taking coordinates $(t, q^i, v^i; \bar{v}^i, w^i)$ in $J^1\bar{\pi}^1$ then $\Upsilon(t, q^i, v^i, w^i) = (t, q^i, v^i; v^i, w^i)$.

Thus, we have the following diagram

$$\begin{tikzcd}
\mathrm{T}J^1\pi = \mathrm{T}\mathbb{R} \times \mathrm{T}(\mathrm{T}Q) \arrow[ddr, "\tau_{J^1\pi}"'] & T^*(J^2\pi) \arrow[d, "\pi_{J^2\pi}"'] & J^1\bar{\pi}^1 = \mathbb{R} \times \mathrm{T}(\mathrm{T}Q) \arrow[ddl, "(\bar{\pi}^1)^1"] \\
 & J^2\pi = \mathbb{R} \times \mathrm{T}^2Q \arrow[ul, "\imath_1"'] \arrow[ur, "\Upsilon"] \arrow[d, "\pi_1^2"'] & \\
 & J^1\pi = \mathbb{R} \times \mathrm{T}Q \arrow[ddl, "\pi^1"'] \arrow[ddr, "\bar{\pi}^1"] & \\
\mathcal{C}_\pi \arrow[r, hook] & \mathrm{T}^*J^1\pi \arrow[u, "\pi_{J^1\pi}"'] & \\
\mathbb{R} \times Q \arrow[rr, "\pi"] & & \mathbb{R}
\end{tikzcd} \tag{34}$$

where the inclusion $\imath_1$ is locally given by $\imath_1(t, q, v, w) = (t, 1, q, v, v, w)$.

Observe that $(\pi_1^2)^*\mathrm{T}^*J^1\pi$ can be identified with a subbundle of $\mathrm{T}^*J^2\pi$ by means of the natural injection $\hat{\imath} \colon (\pi_1^2)^*\mathrm{T}^*J^1\pi \longrightarrow \mathrm{T}^*J^2\pi$, defined as follows: for every $\hat{p} \in J^2\pi$, $\alpha \in \mathrm{T}^*_{\pi_1^2(\hat{p})}J^1\pi$, and $\mathrm{a} \in \mathrm{T}_{\hat{p}}J^2\pi$,

$$(\hat{\imath}(\hat{p}, \alpha))(\mathrm{a}) = \alpha(\mathrm{T}_{\hat{p}}\pi_1^2(\mathrm{a})) \; .$$

In the same way, we can identify $(\pi_1^2)^*\mathcal{C}_\pi$ as a subbundle of $(\pi_1^2)^*\mathrm{T}^*J^1\pi$ by means of $\hat{\imath}$.

Local bases for the set of sections of the bundles $\mathrm{T}^*J^2\pi \longrightarrow J^2\pi$, $(\pi_1^2)^*\mathrm{T}^*J^1\pi \longrightarrow J^2\pi$, and $(\pi_1^2)^*\mathcal{C}_\pi \longrightarrow J^2\pi$ are $(\mathrm{d}t, \mathrm{d}q^i, \mathrm{d}v^i, \mathrm{d}w^i)$, $(\mathrm{d}t, \mathrm{d}q^i, \mathrm{d}v^i)$, and $(\mathrm{d}q^i - v^i\mathrm{d}t)$, respectively.

Incidentally, $Sec\,(J^2\pi, (\pi_1^2)^*\mathrm{T}^*J^1\pi) = \mathrm{C}^\infty(J^2\pi) \otimes_{\mathrm{C}^\infty(J^1\pi)} (\pi_1^2)^*\mathit{\Omega}^1(J^1\pi)$, which are the π_1^2-semibasic 1-forms in $J^2\pi$.

A.3 Euler-Lagrange equations

Let $\mathcal{L} \in \mathit{\Omega}^1(J^1\pi)$ be a Lagrangian density and its associated Lagrangian function $L \in \mathrm{C}^\infty(J^1\pi)$. Observe that

$$d_T\Theta_{\mathcal{L}} \in \mathit{\Omega}^1(\mathrm{T}J^1\pi) \; , \; \imath_1^*d_T\Theta_{\mathcal{L}} \in \mathit{\Omega}^1(J^2\pi) \; , \; (\pi_1^2)^*\mathrm{d}L \in \mathit{\Omega}^1(J^2\pi) \; .$$

Then, a simple calculation in coordinates shows that $\imath_1^*d_T\Theta_{\mathcal{L}} - (\pi_1^2)^*\mathrm{d}L$ is a section of the bundle projection $\hat{\imath}((\pi_1^2)^*\mathcal{C}_\pi) \longrightarrow J^2\pi$.

The Euler-Lagrange equations for this Lagrangian are a system of second order differential equations on Q; that is, in implicit form, a submanifold D of $J^2\pi$ determined by:

$$D = \{\hat{p} \in J^2\pi \mid (\imath_1^*d_T\Theta_{\mathcal{L}} - (\pi_1^2)^*\mathrm{d}L)(\hat{p}) = 0\} = \{\hat{p} \in J^2\pi \mid \mathcal{E}_{\mathcal{L}}(\hat{p}) = 0\} = \mathcal{E}_{\mathcal{L}}^{-1}(0) \; ,$$

where $\mathcal{E}_\mathcal{L} = \imath_1^* d_T\Theta_\mathcal{L} - (\pi_1^2)^*\mathrm{d}L$. Then, a section $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}\times Q$ is a solution to the Lagrangian system if, and only if, $\mathrm{Im}\, j^2\phi \subset \mathcal{E}_\mathcal{L}^{-1}(0)$. In fact, working in local coordinates, such as

$$
\begin{aligned}
d_T\Theta_\mathcal{L} &= \frac{\partial L}{\partial v^k}\mathrm{d}v^k - \left(\frac{\partial L}{\partial v^j}v^j - L\right)\mathrm{d}\dot{t} + \left(\dot{t}\frac{\partial^2 L}{\partial t\partial v^k} + v^i\frac{\partial^2 L}{\partial q^i\partial v^k} + w^i\frac{\partial^2 L}{\partial v^i\partial v^k}\right)\mathrm{d}q^k \\
&\quad - \left[\dot{t}\left(v^j\dot{t}\frac{\partial^2 L}{\partial t\partial v^j} - \frac{\partial L}{\partial t}\right) + v^i\left(v^j\frac{\partial^2 L}{\partial q^i\partial v^j} - \frac{\partial L}{\partial q^i}\right) + w^i\left(\frac{\partial L}{\partial v^i} + v^j\frac{\partial^2 L}{\partial v^i\partial v^j} - \frac{\partial L}{\partial v^i}\right)\right]\mathrm{d}t \\
\imath_1^* d_T\Theta_\mathcal{L} &= \frac{\partial L}{\partial v^k}\mathrm{d}v^k + \left(\frac{\partial^2 L}{\partial t\partial v^k} + v^i\frac{\partial^2 L}{\partial q^i\partial v^k} + w^i\frac{\partial^2 L}{\partial v^i\partial v^k}\right)\mathrm{d}q^k \\
&\quad - \left[v^j\frac{\partial^2 L}{\partial t\partial v^j} - \frac{\partial L}{\partial t} + v^i\left(v^j\frac{\partial^2 L}{\partial q^i\partial v^j} - \frac{\partial L}{\partial q^i}\right) + w^iv^j\frac{\partial^2 L}{\partial v^i\partial v^j}\right]\mathrm{d}t \\
(\pi_1^2)^*\mathrm{d}L &= \frac{\partial L}{\partial t}\mathrm{d}t + \frac{\partial L}{\partial q^k}\mathrm{d}q^k + \frac{\partial L}{\partial v^k}\mathrm{d}v^k \ ,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\imath_1^* d_T\Theta_\mathcal{L} - (\pi_1^2)^*\mathrm{d}L &= \left(\frac{\partial^2 L}{\partial v^i\partial v^k}w^i + \frac{\partial^2 L}{\partial q^i\partial v^k}v^i + \frac{\partial^2 L}{\partial t\partial v^k} - \frac{\partial L}{\partial q^k}\right)(\mathrm{d}q^k - v^k\mathrm{d}t) \\
&= \left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^k}\right) - \frac{\partial L}{\partial q^k}\right](\mathrm{d}q^k - v^k\mathrm{d}t) \ .
\end{aligned}
$$

Now, suppose that there are external forces operating on the Lagrangian system $(J^1\pi, \mathcal{L})$. A force depending on velocities is a section $F\colon J^1\pi \longrightarrow \mathcal{C}_\pi$. As above, the corresponding Euler-Lagrange equations are a system of second order differential equations on Q, given in implicit form by the submanifold D_F of $J^2\pi$ determined by:

$$
D_F = \{\hat{p} \in J^2\pi \mid (\imath_1^* d_T\Theta_\mathcal{L} - (\pi_1^2)^*\mathrm{d}L)(\hat{p}) = (F\circ\pi_1^2)(\hat{p})\} = \{\hat{p}\in J^2\pi \mid \mathcal{E}_\mathcal{L}(\hat{p}) = (F\circ\pi_1^2)(\hat{p})\} \ .
$$

A section $\phi\colon \mathbb{R}\longrightarrow \mathbb{R}\times Q$ is a solution to the Lagrangian system if, and only if,

$$
\mathcal{E}_\mathcal{L}(j^2\phi) = (\pi_1^2)^*[(F\circ\pi_1^2)(j^2\phi)] = (\pi_1^2)^*F(j^1\phi) \ . \tag{35}
$$

In natural coordinates we have

$$
\left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^k}\right) - \frac{\partial L}{\partial q^k}\right](\mathrm{d}q^k - v^k\mathrm{d}t) = F_j(\mathrm{d}q^j - v^j\mathrm{d}t) \ .
$$

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